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Fixed Point Theorems using Absorbing Maps in $\varepsilon -$ Chainable Fuzzy Metric Spaces

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Abstract— The introduction of notion of fuzzy sets by Zadeh [17] and the notion of fuzzy metric spaces by Kramosil and Michalek [9] has led to the extensive development of the theory of fuzzy sets and its applications. Several concepts of analysis and topology have been redefined and extended in fuzzy settings. The numerous applications of fuzzy metric spaces in applied sciences and engineering, particularly in quantum particle physics has prompted many authors to extend the Banach's Contraction Principle to fuzzy metric spaces and prove fixed point and common fixed point theorems for fuzzy metric spaces. In this paper, we prove a common fixed point theorem for six self mappings using the concept of absorbing maps in ε – chainable fuzzy metric space introduced by Cho et al. [2].

Index Terms— Absorbing Maps, Common Fixed Point, ε – Chainable Fuzzy Metric Space, Fuzzy Metric Space, Reciprocal Continuity, Semi - Compatible Maps, Weak Compatibility

1 Introduction

THE notion of fuzzy sets was introduced by Zadeh [17] in 1965. Following this many authors have extensively developed the theory of fuzzy sets and its applications and have redefined and extended several concepts of analysis and topology in fuzzy settings. The idea of a fuzzy metric space was introduced by Kramosil and Michalek [9] which was later modified by George and Veeramani [5]. During the last few decades many authors have established the existence of a lot of fixed point theorems for fuzzy metric spaces, especially Deng zi – ke [3], Erceg [4], George and Veeramani [5 & 6], Grabiec [7], Kaleva and Seikkala [8], Kramosil and Michalek [9], Schwizer and Skalar [14]. Singh and Jain [15] introduced the notion of semi compatible maps in fuzzy metric space and obtained common fixed point theorems for such spaces. Vasuki [16], introduced the concept of R – weakly commuting map and proved a fixed point theorem for fuzzy metric space using this concept. Cho et al [2] introduced the concept of $\varepsilon -$ chainable fuzzy metric space and obtained common fixed point theorems for four weakly compatible mappings of $\varepsilon -$ chainable fuzzy metric spaces. Ranadive et al [13], introduced the concept of absorbing mappings in metric space and proved a common fixed point theorem in this space. Moreover they observed that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non-compatible maps. Mishra et.al [10, 11 & 12] applied the notion of absorbing maps in fuzzy metric spaces and proved a common fixed point theorem in these spaces. In this paper, we prove a common fixed point theorem for six mappings using absorbing maps with $\varepsilon -$ chainable fuzzy metric space. Our paper extends the results of Cho and Jung [1]. For the sake of completeness we recall some definitions and results in the next section.

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2 Preliminaries

Definition 2.1: Let X be a non empty set. Then a function A with domain X and value in $[0,1]$ is said to be a fuzzy set in X.

Definition 2.2: A $t-$norm or more precisely triangular norm $*$ is a binary operation defined on $[0,1]$ such that for all $a, b, c, d \in [0,1]$, following conditions are satisfied:

(1) $a * 1 = 1$;
(2) $a * b = b * a$;
(3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$;
(4) $a * (b * c) = (a * b) * c$.

Definition 2.3: The $3-$tuple $(X, \mathcal{M}, *)$ is called a fuzzy metric space if X is an arbitrary non – empty set, $*$ is a continuous $t-$norm and $\mathcal{M}$ is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(1) $\mathcal{M}(x,y,0) > 0$;
(2) $\mathcal{M}(x,y,t) = 1$ for all $t > 0$, iff $x = y$;
(3) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(4) $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \le \mathcal{M}(x,z,t+s)$;
(5) $\mathcal{M}(x,y,\ .) : [0,\infty) \to [0,1]$ is continuous.

Example 2.1: Let (X,d) be a metric space. Define $a * b = \min(a,b)$, and

$$\mathcal{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

induced by the metric d is often called the standard fuzzy metric.

Definition 2.4: A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}(x_n, x_m, t) > 1 - \varepsilon \text{ for all } n, m \ge n_0.$$

A sequence $\{\mathbf{x_n}\}$ in a fuzzy metric space $(\mathbf{X}, \boldsymbol{\mathcal{M}}, *)$ is said to be convergent to $\mathbf{x} \in \mathbf{X}$ if there exists $\mathbf{n_0} \in \mathbb{N}$ such that $\lim_{n\to\infty} \boldsymbol{\mathcal{M}}(\mathbf{x_n}, \mathbf{x}, \mathbf{t}) > \mathbf{1} - \boldsymbol{\varepsilon}$ for all $\mathbf{t} > \mathbf{0}$ & $\boldsymbol{n} \ge \mathbf{n_0}$. A fuzzy metric space $(\mathbf{X}, \boldsymbol{\mathcal{M}}, *)$ is said to be complete if every Cauchy sequence in $\mathbf{X}$ converges to a point in $\mathbf{X}$.

Definition 2.5: A pair (A,B) of self mappings of a fuzzy metric space $(X,\mathcal{M},*)$ is said to be reciprocal continuous if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} ABx_n = Ax \text{ and } \lim_{n\to\infty} BAx_n = Bx$$

whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$. If A and B are both continuous then they are obviously reciprocally continuous but the converse is not necessarily true.

DEFINITION 2.6: Two self mappings A and B of a fuzzy metric space $(X,\mathcal{M},*)$ are said to be weakly compatible if $ABx = BAx$ whenever $Ax = Bx$ for some $x \in X$.

DEFINITION 2.7: A pair (A,B) of self mappings of a fuzzy metric space $(X,\mathcal{M},*)$ is said to be semi-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} ABx_n = Bx$ whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

DEFINITION 2.8: A finite sequence $x = x_0, x_1, \cdots, x_n = y$ in a fuzzy metric space $(X,\mathcal{M},*)$ is called $\varepsilon -$ chain from x to y if there exists $\varepsilon > 0$ such that $\mathcal{M}(x_i, x_{i-1}, t) > 1 - \varepsilon$ for all $t > 0$ and $i = 1, 2, \cdots, n$.

A fuzzy metric space $(X,\mathcal{M},*)$ is called $\varepsilon -$ chainable if there exists an $\varepsilon -$ chain from x to y, for any $x, y \in X$.

LEMMA 2.1: If for all $x, y \in X$, $t > 0$ and $0 < k < 1$, $\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$, then $x = y$.

PROOF: Suppose that there exists $0 < k < 1$ such that $\mathcal{M}(x,y,kt) \geq \mathcal{M}(x,y,t)$ for all $x, y \in X$ and $t > 0$. Then, $\mathcal{M}(x,y,t) \geq \mathcal{M}\left(x,y,\frac{t}{k}\right)$, and so $\mathcal{M}(x,y,t) \geq \mathcal{M}\left(x,y,\frac{t}{k^n}\right)$ for positive integer n. Taking limit as $n \to \infty$, $\mathcal{M}(x,y,t) \geq 1$ and hence $x = y$.

LEMMA 2.2: $\mathcal{M}(x,y,.)$ is non decreasing for all $x, y \in X$.

PROOF: Suppose $\mathcal{M}(x,y,t) > M(x,y,s)$ for some $0 < t < s$. Then $\mathcal{M}(x,y,t) * \mathcal{M}(y,y,s-t) \leq \mathcal{M}(x,y,s) < M(x,y,t)$. Since $\mathcal{M}(y,y,s-t) = 1$, therefore, $\mathcal{M}(x,y,t) \leq \mathcal{M}(x,y,s) < M(x,y,t)$, which is a contradiction. Thus, $\mathcal{M}(x,y,.)$ is non decreasing for all $x, y \in X$.

DEFINITION 2.9: For two self maps f and g on a fuzzy metric space $(X,\mathcal{M},*)$, f is called $g -$ absorbing if there exists a positive integer $R > 0$ such that $\mathcal{M}(gx, gfx, t) \geq \mathcal{M}\left(gx, fx, \frac{t}{R}\right)$ for all $x \in X$. Similarly, g is called $f -$ absorbing if there exists a positive integer $R > 0$ such that $\mathcal{M}(fx, fgx, t) \geq \mathcal{M}\left(fx, gx, \frac{t}{R}\right)$ for all $x \in X$.

EXAMPLE 2.2: Let (X,d) be the usual metric space with $X = [2, 20]$ and $\mathcal{M}$ be the usual fuzzy metric on a fuzzy metric space $(X,\mathcal{M},*)$ defined by

$$\mathcal{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

and $\mathcal{M}(x,y,0) = 0$ for all $x, y \in X, t > 0$. Define

$$fx = \begin{cases} 6, & \text{if } 2 \leq x \leq 5; \text{ and } x = 6 \\ 10, & \text{if } x > 6 \\ \dfrac{x-1}{2}, & \text{if } 5 < x < 6 \end{cases}$$

$$gx = \begin{cases} 2, & \text{if } 2 \leq x \leq 5 \\ \dfrac{x+1}{3}, & \text{if } x > 5 \end{cases}$$

Then it can be easily verified that both (f,g) and (g,f) are not compatible but f is $g -$ absorbing and g is $f -$ absorbing.

EXAMPLE 2.3: Let (X,d) be the usual metric space with $X = [0, 1]$ and $\mathcal{M}$ be the usual fuzzy metric on a fuzzy metric space $(X,\mathcal{M},*)$ defined by

$$\mathcal{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

and $\mathcal{M}(x,y,0) = 0$ for all $x, y \in X, t > 0$. Define $f, g : X \to X$ by $fx = \frac{x}{16}$ and $gx = 1 - \frac{x}{3}$. Then it can be easily verified that f and g are compatible pair of maps and f is $g -$ absorbing while g is $f -$ absorbing.

THEOREM 2.1: Let $(X,\mathcal{M},*)$ be a complete $\varepsilon -$ chainable fuzzy metric space and let A, B, S and T be self mappings of X satisfying the following conditions:

(1) $AX \subset TX$ and $BX \subset SX$;
(2) A and S are continuous;
(3) The pairs (A,S) and (B,T) are weakly compatible;
(4) There exists $k \in (0,1)$ such that $\mathcal{M}(Ax,By,kt) \geq \mathcal{M}(Sx,Ty,t) * \mathcal{M}(Ax,Sx,t) * \mathcal{M}(By,Ty,t) * \mathcal{M}(Ax,Ty,t)$ for all $x, y \in X$ and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

PROOF: We can find a Cauchy sequence $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2, 3, \cdots$

From completeness, $y_n \to z$ for some $z \in X$, and so $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converge to z. Since X is $\varepsilon -$ chainable, there exists an $\varepsilon -$ chain from x_n to x_{n+1}, that is, there exists a finite sequence $x_n = y_1, y_2, \cdots, y_l = x_{n+1}$ such that $\mathcal{M}(y_i, y_{i-1}, t) > 1 - \varepsilon$ for all $t > 0$ and $i = 1, 2, \cdots, l$. Thus we have

$$\begin{aligned}\mathcal{M}(x_n, x_{n+1}, t) &\geq \mathcal{M}\left(y_1, y_2, \frac{t}{l}\right) * \mathcal{M}\left(y_2, y_3, \frac{t}{l}\right) * \cdots * \mathcal{M}\left(y_{l-1}, y_l, \frac{t}{l}\right) \\ &> (1-\varepsilon) * (1-\varepsilon) * \cdots * (1-\varepsilon) \\ &\geq (1-\varepsilon).\end{aligned}$$

For $m > n$,

$$\begin{aligned}\mathcal{M}(x_n, x_m, t) &\geq \mathcal{M}\left(x_n, x_{n+1}, \frac{t}{m-n}\right) * \mathcal{M}\left(x_{n+1}, x_{n+2}, \frac{t}{m-n}\right) * \cdots * \mathcal{M}\left(x_{m-1}, x_m, \frac{t}{m-n}\right) \\ &> (1-\varepsilon) * (1-\varepsilon) * \cdots * (1-\varepsilon) \\ &> (1-\varepsilon),\end{aligned}$$

From (2), $Ax_{2n-2} \to Ax$ and $Sx_{2n} \to Sx$. Since X is Hausdorff, $Ax = z = Sx$. Because (A,S) is weakly compatible $ASx = SAx$ and so $Az = Sz$. From (2), $ASx_{2n} \to ASx$ and so $ASx_{2n} \to Sz$. Also, from continuity of S, $SSx_{2n} \to Sz$. From (4),

$$\begin{aligned}\mathcal{M}(ASx_{2n}, Bx_{2n-1}, kt) \geq\ & \mathcal{M}(SSx_{2n}, Tx_{2n-1}, t) * \mathcal{M}(ASx_{2n}, SSx_{2n}, t) \\ & * \mathcal{M}(Bx_{2n-1}, Tx_{2n-1}, t) * \mathcal{M}(ASx_{2n}, Tx_{2n-1}, t)\end{aligned}$$

Taking limit as $n \to \infty$,

$$\begin{aligned}\mathcal{M}(Sz, z, kt) &\geq \mathcal{M}(Sz, z, t) * \mathcal{M}(Sz, Sz, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Sz, z, t) \\ &\geq \mathcal{M}(Sz, z, t).\end{aligned}$$

Thus $Sz = z$, and hence $Az = Sz = z$. Since $AX \subset TX$, there exists $v \in X$ such that $Tv = Az = z$. From (4),

$$\mathcal{M}(Ax_{2n}, Bv, kt) \geq \mathcal{M}(Sx_{2n}, Tv, t) * \mathcal{M}(Ax_{2n}, Sx_{2n}, t) * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(Ax_{2n}, Tv, t)$$

Letting $n \to \infty$, we have

$$\mathcal{M}(z, Bv, kt) \geq \mathcal{M}(z, Tv, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bv, Tv, t) * \mathcal{M}(z, Tv, t)$$

$= \mathcal{M}(z,z,t) * \mathcal{M}(z,z,t) * \mathcal{M}(Bv,z,t) * \mathcal{M}(z,z,t)$
$\geq \mathcal{M}(Bv,z,t)$

and so $Bv = z$ and hence $Tv = Bv = z$. Since (B,T) is weakly compatible,

$TBv = BTv$ and hence $Tz = Bz$. From (4)

$\mathcal{M}(Ax_{2n},Bz,kt) \geq \mathcal{M}(Sx_{2n},Tz,t) * \mathcal{M}(Ax_{2n},Sx_{2n},t) * \mathcal{M}(Bz,Tz,t) * \mathcal{M}(Ax_{2n},Tz,t)$

Taking limit as $n \to \infty$,

$$\mathcal{M}(z,Bz,kt) \geq \mathcal{M}(z,Tz,t) * \mathcal{M}(z,z,t) * \mathcal{M}(Bz,Tz,t) * \mathcal{M}(z,Tz,t) = \mathcal{M}(z,Bz,t) * \mathcal{M}(z,z,t) * \mathcal{M}(Bz,Bz,t) * \mathcal{M}(z,Bz,t) \geq \mathcal{M}(z,Bz,t)$$

which implies that $Bz = z$. From $Az = Sz = z$, $Tz = Bz$ and $Bz = z$, it follows that A, B, S and T have a common fixed point z in X.

For uniqueness, let w be another common fixed point of A, B, S and T. Then

$$\mathcal{M}(z,w,kt) = \mathcal{M}(Az,Bw,kt) \geq \mathcal{M}(Sz,Tw,t) * \mathcal{M}(Az,Sz,t) * \mathcal{M}(Bw,Tw,t) * \mathcal{M}(Az,Tw,t) \geq \mathcal{M}(z,w,t).$$

Thus $z = w$.

We now establish the following theorem.

3 MAIN THEOREM

THEOREM 3.1: Let A, B, S, T, P and Q be self mappings of a complete $\epsilon-$chainable fuzzy metric space $(X,\mathcal{M},*)$ with continuous $t-$norm satisfying the conditions:

1. $P(X) \subseteq ST(X), Q(X) \subseteq AB(X)$
2. $AB = BA, ST = TS, PB = BP, QT = TQ$
3. Q is $ST-$absorbing
4. There exists $k \in (0,1)$, such that
   $\mathcal{M}(Px,Qy,kt) \geq \min\{\mathcal{M}(ABx,STy,t), \mathcal{M}(Px,ABx,t), \frac{1}{2}(\mathcal{M}(ABx,Qy,t) + \mathcal{M}(Px,STy,t)), \mathcal{M}(STy,Qy,t)\}$

for every $x,y \in X$ and $t > 0$. If (P,AB) is reciprocally continuous semi compatible maps, then A, B, S, T, P and Q have unique common fixed point in X.

PROOF: Let $x_0 \in X$ then from (1) there exist $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. In general we can find a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0,1,2,\dots$. Putting $x = x_{2n+2}, y = x_{2n+1}$ for all $t > 0$ in condition (4) we have:

$$\mathcal{M}(y_{2n+1},y_{2n+2},kt) = \mathcal{M}(Px_{2n+2},Qx_{2n+1},kt)$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(ABx_{2n+2},STx_{2n+1},t), \mathcal{M}(Px_{2n+2},ABx_{2n+2},t),\\ \frac{1}{2}(\mathcal{M}(ABx_{2n+2},Qx_{2n+1},t) + \mathcal{M}(Px_{2n+2},STx_{2n+1},t)),\\ \mathcal{M}(STx_{2n+1},Qx_{2n+1},t)\end{array}\right\}$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(y_{2n+1},y_{2n},t), \mathcal{M}(y_{2n+2},y_{2n+1},t),\\ \frac{1}{2}(\mathcal{M}(y_{2n+1},y_{2n+1},t) + \mathcal{M}(y_{2n+2},y_{2n},t)),\\ \mathcal{M}(y_{2n},y_{2n+1},t)\end{array}\right\}$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(y_{2n+1},y_{2n},t), \mathcal{M}(y_{2n+2},y_{2n+1},t),\\ \frac{1}{2}(1 + \mathcal{M}(y_{2n+2},y_{2n},t)), \mathcal{M}(y_{2n},y_{2n+1},t)\end{array}\right\}$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(y_{2n+1},y_{2n},t), \mathcal{M}(y_{2n+2},y_{2n+1},t), \mathcal{M}(y_{2n+2},y_{2n},t),\\ \mathcal{M}(y_{2n},y_{2n+1},t)\end{array}\right\}$$

Hence, $\mathcal{M}(y_{2n+1},y_{2n+2},kt) \geq \mathcal{M}(y_{2n+1},y_{2n},t)$.

Again, putting $x = x_{2n+2}, y = x_{2n+3}$ for all $t > 0$ in condition (4) we have:

$$\mathcal{M}(y_{2n+2},y_{2n+3},kt) = \mathcal{M}(Px_{2n+2},Qx_{2n+3},kt)$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(ABx_{2n+2},STx_{2n+3},t), \mathcal{M}(Px_{2n+2},ABx_{2n+2},t),\\ \frac{1}{2}(\mathcal{M}(ABx_{2n+2},Qx_{2n+3},t) + \mathcal{M}(Px_{2n+2},STx_{2n+3},t)),\\ \mathcal{M}(STx_{2n+3},Qx_{2n+3},t)\end{array}\right\}$$
$$\geq \min\{\mathcal{M}(y_{2n+1},y_{2n+2},t), \mathcal{M}(y_{2n+2},y_{2n+1},t), \frac{1}{2}(\mathcal{M}(y_{2n+1},y_{2n+3},t) + \mathcal{M}(y_{2n+2},y_{2n+2},t)), \mathcal{M}(y_{2n+2},y_{2n+3},t)\}$$
$$\geq \min\{\mathcal{M}(y_{2n+1},y_{2n+2},t), \mathcal{M}(y_{2n+2},y_{2n+1},t), \frac{1}{2}(\mathcal{M}(y_{2n+1},y_{2n+3},t) + 1), \mathcal{M}(y_{2n+2},y_{2n+3},t)\}$$
$$\geq \min\left\{\begin{array}{c}\mathcal{M}(y_{2n+1},y_{2n+2},t), \mathcal{M}(y_{2n+2},y_{2n+1},t), \mathcal{M}(y_{2n+1},y_{2n+3},t),\\ \mathcal{M}(y_{2n+2},y_{2n+3},t)\end{array}\right\}$$

Hence, $\mathcal{M}(y_{2n+2},y_{2n+3},kt) \geq \mathcal{M}(y_{2n+1},y_{2n+2},t)$.

Therefore for all n, we have

$$\mathcal{M}(y_n,y_{n+1},t) \geq \mathcal{M}(y_n,y_{n-1},t/k) \geq \mathcal{M}(y_n,y_{n-1},t/k^2) \geq \cdots \geq \mathcal{M}(y_n,y_{n-1},t/k^n)$$

i.e. $\mathcal{M}(y_n,y_{n+1},t) \to 1$ as $n \to \infty$,

for any $t > 0$. For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that $\mathcal{M}(y_n,y_{n+1},t) > 1 - \varepsilon$ for all $n > n_0$. For $m,n \in \mathbb{N}$, we suppose $m \geq n$. Then we have that

$$\mathcal{M}(y_n,y_m,t) \geq \mathcal{M}(y_n,y_{n+1},t/m-n) * \mathcal{M}(y_{n+1},y_{n+2},t/m-n) * \cdots * \mathcal{M}(y_{m-1},y_m,t/m-n) > (1-\varepsilon) * (1-\varepsilon) * \cdots * (1-\varepsilon) \geq (1-\varepsilon).$$

Hence $\{y_n\}$ is a Cauchy sequence in X; that is $y_n \to z$ in X; so its subsequences Px_{2n}, STx_{2n+1}, ABx_{2n}, Qx_{2n+1} also converge to z. Since X is $\epsilon-$chainable, there exists $\epsilon-$chain from x_n to x_{n+1}, that is there exists a finite sequence $x_n = y_1, y_2, \cdots, y_l = x_{n+1}$, such that

$\mathcal{M}(y_i,y_{i-1},t) > (1-\varepsilon)$ for all $t > 0$ and $i = 1,2,\cdots,l$. Thus we have

$$\mathcal{M}(x_n,x_{n+1},t) > \mathcal{M}(y_1,y_2,t/l) * \mathcal{M}(y_2,y_3,t/l) * \cdots * \mathcal{M}(y_{i-1},y_i,t/l)$$

$> (1-\varepsilon) * (1-\varepsilon) * (1-\varepsilon) * \cdots * (1-\varepsilon) \geq (1-\varepsilon)$, and so $\{x_n\}$ is a Cauchy sequence in X and hence there exists $z \in X$ such that $x_n \to z$. Since the pair of (P,AB) is reciprocal continuous; we have $\lim_{n\to\infty} P(AB)x_{2n} \to Pz$ and $\lim_{n\to\infty} AB(P)x_{2n} \to ABz$ and the semi compatibility of (P,AB) gives $\lim_{n\to\infty} AB(P)x_{2n} \to ABz$, therefore $Pz = ABz$. We claim

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$Pz = ABz = z.$

Step I: Putting $x = z$ and $y = x_{2n+1}$ in condition (4) we have

$$\mathcal{M}(Pz, Qx_{2n+1}, kt) \geq \min\{\mathcal{M}(ABz, STx_{2n+1}, t), \mathcal{M}(Pz, ABz, t), \frac{1}{2}\big(\mathcal{M}(ABz, Qx_{2n+1}, t) + \mathcal{M}(Pz, STx_{2n+1}, t)\big), \mathcal{M}(STx_{2n+1}, Qx_{2n+1}, t)\}$$

Letting $n \to \infty$ we have

$$\mathcal{M}(Pz, z, kt) \geq \min\{\mathcal{M}(Pz, z, t), \mathcal{M}(Pz, Pz, t), \frac{1}{2}\big(\mathcal{M}(Pz, z, t) + \mathcal{M}(Pz, z, t)\big), \mathcal{M}(z, z, t)\}$$

$$\geq \min\{\mathcal{M}(Pz, z, t), \mathcal{M}(Pz, Pz, t), \mathcal{M}(Pz, z, t), \mathcal{M}(z, z, t)\}$$

i.e. $\mathcal{M}(Pz, z, kt) \geq \mathcal{M}(Pz, z, t)$

Therefore $z = Pz = ABz.$

Step II: Putting $x = Bz$ and $y = x_{2n+1}$ in condition (4) we have

$$\mathcal{M}(P(Bz), Qx_{2n+1}, kt) \geq \min\left\{\begin{array}{c}\mathcal{M}(AB(Bz), STx_{2n+1}, t), \mathcal{M}(P(Bz), AB(Bz), t), \\ \frac{1}{2}\big(\mathcal{M}(AB(Bz), Qx_{2n+1}, t) + \mathcal{M}(P(Bz), STx_{2n+1}, t)\big), \\ \mathcal{M}(STx_{2n+1}, Qx_{2n+1}, t)\end{array}\right\}$$

Since $PB = BP, AB = BA$ so $P(Bz) = B(Pz) = Bz$ and $AB(Bz) = B(ABz) = Bz$

Letting $n \to \infty$ we have

$$\mathcal{M}(Bz, z, kt) \geq \min\{\mathcal{M}(Bz, z, t), \mathcal{M}(Bz, Bz, t), \frac{1}{2}\big(\mathcal{M}(Bz, z, t) + \mathcal{M}(Bz, z, t)\big), \mathcal{M}(z, z, t)\}$$

$$\geq \min\{\mathcal{M}(Bz, z, t), \mathcal{M}(Bz, Bz, t), \mathcal{M}(Bz, z, t), \mathcal{M}(z, z, t)\}$$

i.e. $\mathcal{M}(Bz, z, kt) \geq \mathcal{M}(Bz, z, t)$

Therefore, $Az = Bz = Pz = z.$

Step III: Since $P(X) \subseteq ST(X)$, there exists $u \in X$ such that $z = Pz = STu$.

Putting $x = x_{2n}$, $y = u$ in condition (4) we have

$$\mathcal{M}(Px_{2n}, Qu, kt) \geq \min\{\mathcal{M}(ABx_{2n}, STu, t), \mathcal{M}(Px_{2n}, ABx_{2n}, t), \frac{1}{2}\big(\mathcal{M}(ABx_{2n}, Qu, t) + \mathcal{M}(Px_{2n}, STu, t)\big), \mathcal{M}(STu, Qu, t)\}$$

Letting $n \to \infty$ we have

$$\mathcal{M}(z, Qu, kt) \geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \frac{1}{2}\big(\mathcal{M}(z, Qu, t) + \mathcal{M}(z, z, t)\big), \mathcal{M}(z, Qu, t)\}$$

$$\geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \frac{1}{2}(\mathcal{M}(z, Qu, t) + 1), \mathcal{M}(z, Qu, t)\}$$

$$\geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \mathcal{M}(z, Qu, t), \mathcal{M}(z, Qu, t)\}$$

i.e. $\mathcal{M}(z, Qu, kt) \geq \mathcal{M}(z, Qu, t)$

Therefore, $z = Qu = STu.$

Since Q is $ST-$absorbing, therefore

$$\mathcal{M}(STu, STQu, kt) \geq \mathcal{M}(STu, Qu, {t}/{R}) = 1$$

i.e. $STu = STQu \Longrightarrow z = STz.$

Step IV: Putting $x = x_{2n}$, $y = z$ in condition (4) we have

$$\mathcal{M}(Px_{2n}, Qz, kt) \geq \min\{\mathcal{M}(ABx_{2n}, STz, t), \mathcal{M}(Px_{2n}, ABx_{2n}, t), \frac{1}{2}\big(\mathcal{M}(ABx_{2n}, Qz, t) + \mathcal{M}(Px_{2n}, STz, t)\big), \mathcal{M}(STz, Qz, t)\}$$

Letting $n \to \infty$ we have

$$\mathcal{M}(z, Qz, kt) \geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \frac{1}{2}\big(\mathcal{M}(z, Qz, t) + \mathcal{M}(z, z, t)\big), \mathcal{M}(z, Qz, t)\}$$

$$\geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \frac{1}{2}(\mathcal{M}(z, Qz, t) + 1), \mathcal{M}(z, Qz, t)\}$$

$$\geq \min\{\mathcal{M}(z, z, t), \mathcal{M}(z, z, t), \mathcal{M}(z, Qz, t), \mathcal{M}(z, Qz, t)\}$$

i.e. $\mathcal{M}(z, Qz, kt) \geq \mathcal{M}(z, Qz, t)$

Therefore, $z = Qz = STz.$

Step V: Putting $x = x_{2n}$, $y = Tz$ in condition (4) we have

$$\mathcal{M}(Px_{2n}, QTz, kt) \geq \min\{\mathcal{M}(ABx_{2n}, ST(Tz), t), \mathcal{M}(Px_{2n}, ABx_{2n}, t), \frac{1}{2}\big(\mathcal{M}(ABx_{2n}, QTz, t) + \mathcal{M}(Px_{2n}, ST(Tz), t)\big), \mathcal{M}(ST(Tz), QTz, t)\}$$

Since $QT = TQ$ & $ST = TS$ therefore

$$QTz = T(Qz) = Tz \;\&\; ST(Tz) = T(STz) = Tz$$

Letting $n \to \infty$ we have

$$\mathcal{M}(z, Tz, kt) \geq \min\{\mathcal{M}(z, Tz, t), \mathcal{M}(z, z, t), \frac{1}{2}\big(\mathcal{M}(z, Tz, t) + \mathcal{M}(z, Tz, t)\big), \mathcal{M}(Tz, Tz, t)\}$$

$$\geq \min\{\mathcal{M}(z, Tz, t), \mathcal{M}(z, z, t), \mathcal{M}(z, Tz, t), \mathcal{M}(Tz, Tz, t)\}$$

i.e. $\mathcal{M}(z, Tz, kt) \geq \mathcal{M}(z, Tz, t)$

Therefore, $z = Tz = Sz = Qz.$

Hence, $z = Az = Bz = Pz = Sz = Qz = Tz.$

Uniqueness: Let w be another fixed point of A, B, P, S, Q & T. Putting $x = u, y = w$ in condition (4), we have

$$\mathcal{M}(Pu, Qw, kt) \geq \min\{\mathcal{M}(ABu, STw, t), \mathcal{M}(Pu, ABu, t), \frac{1}{2}\big(\mathcal{M}(ABu, Qw, t) + \mathcal{M}(Pu, STw, t)\big), \mathcal{M}(STw, Qw, t)\}$$

$$\geq \min\{\mathcal{M}(u, w, t), \mathcal{M}(u, u, t), \frac{1}{2}\big(\mathcal{M}(u, w, t) + \mathcal{M}(u, w, t)\big), \mathcal{M}(w, w, t)\}$$

$$\geq \min\{\mathcal{M}(u, w, t), \mathcal{M}(u, u, t), \mathcal{M}(u, w, t), \mathcal{M}(w, w, t)\}$$

i.e. $\mathcal{M}(\mathrm{u},\mathrm{w},\mathrm{kt}) \geq \mathcal{M}(\mathrm{u},\mathrm{w},\mathrm{t})$

Hence, $\mathrm{z} = \mathrm{w}$.

4 CONCLUSION

In recent years fuzzy fixed point theory has drawn the attention of specialists in fixed point theory and has become their area of interest because of the wide applications of Fuzzy set theory and Fuzzy Fixed Point Theory in applied sciences and engineering such as neural network theory, stability theory, mathematical programming, modeling theory, medical sciences (medical genetics, nervous system), image processing, control theory, communications etc. In this paper we have extended the results of Cho and Jung [1] and proved the existence and uniqueness of a common fixed point for six mappings using the concept of absorbing maps in $\varepsilon -$chainable fuzzy metric spaces.

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